

VOLTERRA OPERATORS SIMILAR TO

$$J: f(x) \rightarrow \int_0^x f(y) dy$$

BY

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Introduction. In this paper we study perturbations $J + P$ of the Volterra operator

$$J: f(x) \rightarrow \int_0^x f(y) dy$$

on $L^r(0, 1)$ ($1 \leq r \leq \infty$). Sufficient conditions—which are in a precise sense sharp—will be obtained for the similarity of J and $J + P$, where P is also a Volterra operator

$$P: f(x) \rightarrow \int_0^x p(x, y)f(y) dy.$$

As an important biproduct of this result it follows immediately that the lattices of invariant subspaces of certain Volterra operators are isomorphic to the (known) lattice of invariant subspaces of J ⁽²⁾. But beyond the result itself and its corollaries, it is interesting to compare the methods used here with those of [4] where we were concerned with the question of similarity to the unilateral shift operator.

As in [4], we rewrite the similarity equation $J = X^{-1}(J + P)X$ as a derivational equation

$$(1) \quad \Delta X = -PX,$$

where Δ is the derivation $\Delta X = JX - XJ$. Guided now by the analogy with the classical differential equation $dX(t)/dt = -P(t)X(t)$ in a Banach algebra (see [8]), we pass to an “integral equation”

$$(2) \quad X = I - \Gamma(PX).$$

Here Γ is an integral—yet to be defined—corresponding to the derivation Δ :

$$(3) \quad \Delta \Gamma(Q) = Q$$

for all Q in a space $\mathcal{L}$ of “integrable” operators. Any solution X of (2) will then also solve (1).

The burden of the method lies in the determination of the integral Γ .

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⁽²⁾ A detailed discussion of this question is found in Kalisch [5].

More precisely, spaces $\mathcal{L}$ and $\mathcal{A}$ of operators and a linear mapping $\Gamma: \mathcal{L} \rightarrow \mathcal{A}$ are defined satisfying (3) and the crucial condition

$$(4) \quad \mathcal{L} \mathcal{A} \subset \mathcal{L}.$$

Then, given $P \in \mathcal{L}$, a solution of (2) will have the form $X = I + \Gamma(Q)$ with $Q \in \mathcal{L}$. Substitution in (1) yields the integral equation

$$(5) \quad Q + P\Gamma(Q) = -P$$

for Q . Conversely, a simple calculation shows that if Q solves (5), then $X = I + \Gamma(Q)$ is a solution of (2).

Thus in the abstract setting described above (and to be realized in what follows for the operator J), the equations (2) and (5) are equivalent. In [4] attention was focused on an equation of the form (2) whose solution was expressible as a discrete product integral. Here, instead, we choose to work with (5). This has the advantage, among others, of avoiding the artificial adjunction of an identity to certain spaces of kernels⁽³⁾. By what amounts to a successive approximations procedure, (5) will be shown to have a solution Q . Then $X = I + \Gamma(Q)$ is a nonsingular solution of (1) and hence implements the similarity $J \sim J + P$.

1. Preliminaries. Given Volterra operators $K: f(x) \rightarrow \int_0^x k(x, y)f(y) dy$ and $L: f(x) \rightarrow \int_0^x l(x, y)f(y) dy$, then (under mild restrictions on the kernels k and l) KL is the Volterra operator

$$KL: f(x) \rightarrow \int_0^x k * l(x, y)f(y) dy,$$

where

$$(6) \quad k * l(x, y) = \int_y^x k(x, \eta)l(\eta, y) d\eta.$$

Instead of dealing with the operators directly we will work with spaces of kernels and the composition $k * l$. To begin with, by 'kernel' we will simply mean a (measurable) complex-valued function $k(x, y)$ on $0 \leq y < x \leq 1$. For $\alpha > 0$ we define

$$(7) \quad \|k\|_{\alpha, \infty} = \text{ess-sup}_{0 \leq y < x \leq 1} |k(x, y)(x - y)^{1-\alpha}|$$

(where ess-sup denotes the essential supremum with respect to Lebesgue measure).

In order to relate $\|k\|_{\alpha, \infty}$ and $\|K\|_r$, the operator norm of

$$K: f(x) \rightarrow \int_0^x k(x, y)f(y) dy$$

⁽³⁾ (5) might well be thought of as the Wiener-Hopf form of (2). In this connection, see [1], [2].

on $L'(0, 1)$, we need the following version of a theorem of M. Riesz (see [3, p. 518]).

THEOREM 1.1. *Let (S, Σ, μ) be a positive measure space and k a measurable function on $S \times S$ with*

$$\operatorname{ess-sup}_x \int_S |k(x, y)| \mu(dy) \leq M < \infty,$$

and

$$\operatorname{ess-sup}_y \int_S |k(x, y)| \mu(dx) \leq M.$$

Then $K: f(x) \rightarrow \int_S k(x, y)f(y)\mu(dy)$ is a bounded operator on $L'(S, \Sigma, \mu)$ ($1 \leq r \leq \infty$) with $\|K\|_r \leq M$.

LEMMA 1.2. *If $\|k\|_{\alpha, \infty} < \infty$, then $K: f(x) \rightarrow \int_0^x k(x, y)f(y) dy$ is a bounded operator on $L'(0, 1)$ ($1 \leq r \leq \infty$) and*

$$\|K\|_r \leq \frac{1}{\alpha} \|k\|_{\alpha, \infty}.$$

Proof. We have, immediately from (7)

$$\operatorname{ess-sup}_{0 \leq x \leq 1} \int_0^x |k(x, y)| dy \leq \|k\|_{\alpha, \infty} \sup_{0 \leq x \leq 1} \int_0^x \frac{dy}{(x-y)^{1-\alpha}}$$

and

$$\operatorname{ess-sup}_{0 \leq y \leq 1} \int_y^1 |k(x, y)| dx \leq \|k\|_{\alpha, \infty} \sup_{0 \leq y \leq 1} \int_y^1 \frac{dx}{(x-y)^{1-\alpha}}$$

and hence, by Theorem 1.1, we get $\|K\|_r \leq C \|k\|_{\alpha, \infty}$ with $C = \int_0^1 (x^{\alpha-1})^{-1} dx = 1/\alpha$.

LEMMA 1.3. *If k and l are kernels for which $\|k\|_{\alpha, \infty} < \infty$ and $\|l\|_{\beta, \infty} < \infty$, then*

$$\|k * l\|_{\alpha+\beta, \infty} \leq B(\alpha, \beta) \|k\|_{\alpha, \infty} \|l\|_{\beta, \infty}$$

(where $B(\alpha, \beta)$ is the beta function).

Proof. Since $|k(x, \eta)l(\eta, y)| \leq (\|k\|_{\alpha, \infty} \|l\|_{\beta, \infty}) / ((x-\eta)^{1-\alpha}(\eta-y)^{1-\beta})$ essentially, it follows that (essentially),

$$\begin{aligned} |k * l(x, y)| &\leq \|k\|_{\alpha, \infty} \|l\|_{\beta, \infty} \int_y^x \frac{d\eta}{(x-\eta)^{1-\alpha}(\eta-y)^{1-\beta}} \\ &= \|k\|_{\alpha, \infty} \|l\|_{\beta, \infty} (x-y)^{\alpha+\beta-1} \int_0^1 \frac{dt}{t^{1-\alpha}(1-t)^{1-\beta}}. \end{aligned}$$

Since the last integral is $B(\alpha, \beta)$, this is equivalent to the asserted inequality.

At this point we remark that all kernels encountered here will have $\|k\|_{\alpha, \infty} < \infty$ for some $\alpha > 0$ and the correspondence mentioned above between the composition $*$ and composition of operators is valid. Likewise associativity holds; $(k * l) * m = k * (l * m)$. We write

$$k^{(n)} = k * k * \dots * k \quad (n\text{-factors}).$$

For example, we have for the iterates of J ,

$$J^n: f(x) \rightarrow \int_0^x \mathbf{1}^{(n)}(x, y) f(y) dy,$$

where

$$\mathbf{1}^{(n)}(x, y) = \frac{(x - y)^{n-1}}{(n-1)!}.$$

LEMMA 1.4. *If $\|k\|_{\alpha, \infty} < \infty$, then*

$$\|k^{(n)}\|_{n\alpha, \infty} \leq \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} \|k\|_{\alpha, \infty}^n$$

(where Γ denotes the gamma function).

Proof. This holds for $n = 1$. Assuming inductively that it holds for n , we have, by Lemma 1.3,

$$\begin{aligned} \|k^{(n+1)}\|_{(n+1)\alpha, \infty} &= \|k^{(n)} * k\|_{(n+1)\alpha, \infty} \leq B(n\alpha, \alpha) \|k^{(n)}\|_{n\alpha, \infty} \|k\|_{\alpha, \infty} \\ &\leq B(n\alpha, \alpha) \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} \|k\|_{\alpha, \infty}^{n+1} = \frac{\Gamma(\alpha)^{n+1}}{\Gamma((n+1)\alpha)} \|k\|_{\alpha, \infty}^{n+1}. \end{aligned}$$

The last equality follows from the identity $B(\gamma, \alpha) = \Gamma(\gamma)\Gamma(\alpha)/\Gamma(\gamma + \alpha)$.

LEMMA 1.5. *If $\|k\|_{\alpha, \infty} < \infty$, then the norms of the operators*

$$K^n: f(x) \rightarrow \int_0^x k^{(n)}(x, y) f(y) dy$$

satisfy

$$\|K^n\|_r \leq \frac{\Gamma(\alpha)^n}{\Gamma((n+1)\alpha)} \|k\|_{\alpha, \infty}^n.$$

Thus $\lim \|K^n\|_r^{1/n} = 0$, i.e., K is a quasi-nilpotent operator on $L^r(0, 1)$, and hence $I + \lambda K$ is nonsingular for every complex number λ .

Proof. By Lemma 1.2, $\|K^n\|_r \leq (1/\alpha) \|k^{(n)}\|_{n\alpha, \infty}$. By the preceding lemma, this in turn is majorized by $\Gamma(\alpha)^n \|k\|_{\alpha, \infty}^n / \alpha \Gamma(n\alpha)$. That $\lim \|K^n\|_r^{1/n} = 0$ now follows since $\lim \Gamma(n\alpha)^{1/n} = \infty$ when $\alpha > 0$.

LEMMA 1.6. *If kernels k and l are continuous on $0 \leq y < x \leq 1$ and $\|k\|_{\alpha, \infty}, \|l\|_{\beta, \infty} < \infty$, then $k * l$ is continuous on $0 \leq y < x \leq 1$. If $\alpha + \beta > 1$, then $k * l$ is continuous on $0 \leq y \leq x \leq 1$ with $k * l(x, x) \equiv 0$.*

Proof. By the assumption, $k(x, y) = m(x, y)/(x - y)^{1-\alpha}$ and $l(x, y) = n(x, y)/(x - y)^{1-\beta}$, where m and n are continuous and bounded on $0 \leq y < x \leq 1$. When $y < x$ the variable change $\eta = y + t(x - y)$ gives

$$k * l(x, y) = (x - y)^{\alpha+\beta-1} \int_0^1 \frac{m[x, y + t(x - y)] n[y + t(x - y), y]}{(1 - t)^{1-\alpha} t^{1-\beta}} dt.$$

Denote the integrand above by $f_{(x,y)}(t)$. If $0 \leq y_0 < x_0 \leq 1$ and (x, y) converges to (x_0, y_0) , then the number $k * l(x, y)$ converges to $k * l(x_0, y_0)$, by the dominated convergence theorem. For $f_{(x,y)}(t)$ converges to $f_{(x_0, y_0)}(t)$ when $0 < t < 1$ and $|f_{(x,y)}(t)| \leq \text{const}/(1 - t)^{1-\alpha} t^{1-\beta}$. That $k * l(x, y)$ converges to 0 as (x, y) converges to (x_0, x_0) follows also from the above expression for $k * l$ providing $\alpha + \beta > 1$.

LEMMA 1.7. If $k(x, y)$ is continuous on $0 \leq x \leq y \leq 1$, $k_1(x, y) = \partial k(x, y)/\partial x$ and $l(x, y)$ are continuous on $0 \leq y < x \leq 1$, and $\|k_1\|_{\alpha, \infty}, \|l\|_{\beta, \infty} < \infty$ for some $\alpha, \beta > 0$ then,

$$\frac{\partial}{\partial x} k * l(x, y) = k_1 * l(x, y) + k(x, x)l(x, y).$$

Proof. For $0 \leq y < x \leq 1$,

$$\begin{aligned} & \frac{1}{h} [k * l(x + h, y) - k * l(x, y)] \\ &= \int_y^x \frac{k(x + h, \eta) - k(x, \eta)}{h} l(\eta, y) d\eta + \frac{1}{h} \int_x^{x+h} k(x, \eta) l(\eta, y) d\eta \\ &+ \int_x^{x+h} \frac{k(x + h, \eta) - k(x, \eta)}{h} l(\eta, y) d\eta. \end{aligned}$$

As $h \rightarrow 0$, the first integral converges to $k_1 * l(x, y)$ by dominated convergence, the second to $k(x, x)l(x, y)$ by continuity of the integrand (recalling that $y < x$), and the third to 0 since the integrand is integrable, uniformly in h , in an interval about x .

We continue to use the subscript 1 to denote differentiation with respect to x . From Lemmas 1.6 and 1.7 we have the following

PROPOSITION 1.8. If a is continuous on $0 \leq y < x \leq 1$ and $\|a\|_{\alpha, \infty} < \infty$ then $1 * a$ and $1^{(2)} * a$ are continuous on $0 \leq y \leq x \leq 1$ and vanish identically on the diagonal. Moreover, $(1^{(2)} * a)_1 = 1 * a$ and $(1^{(2)} * a)_{11} = a$. Conversely, if q is such that

- (i) q and q_1 are continuous on $0 \leq y \leq x \leq 1$,
- (ii) $q(x, x) = q_1(x, x) \equiv 0$ on $0 \leq x \leq 1$, and
- (iii) q_{11} exists and is continuous on $0 \leq y < x \leq 1$
and $\|q_{11}\|_{\alpha, \infty} < \infty$ for some $\alpha > 0$,

then $q_1 = 1 * q_{11}$ and $q = 1^{(2)} * q_{11}$.

2. The "integral" Γ . In this section we are concerned with the integration of the derivational equation $\Delta X = Q$ where Q is a Volterra operator. If we assume a solution $X = \Gamma(Q)$ of the form

$$\Gamma(Q) : f(x) \rightarrow \int_0^x \Gamma(q)(x, y) f(y) dy,$$

then in terms of the kernels, the equation becomes

$$(8) \quad 1 * \Gamma(q) - \Gamma(q) * 1 = q.$$

Let $\mathcal{A}_\alpha$ be the class of kernels which are continuous on $0 \leq y < x \leq 1$ and for which $\|a\|_{\alpha, \infty} < \infty$. We set

$$(9) \quad \mathcal{L}_\alpha = 1^{(2)} * \mathcal{A}_\alpha,$$

i.e., $\mathcal{L}_\alpha$ is the class of kernels of the form $q = 1^{(2)} * a$ with $a \in \mathcal{A}_\alpha$. It follows by Lemmas 1.3 and 1.6 that the spaces $\mathcal{A}_\alpha$ and $\mathcal{L}_\alpha$ decrease as α increases and satisfy the relations

$$(10) \quad \begin{aligned} \mathcal{A}_\alpha * \mathcal{A}_\beta &\subset \mathcal{A}_{\alpha+\beta}, \\ \mathcal{L}_\alpha * \mathcal{A}_\beta &\subset \mathcal{L}_{\alpha+\beta}. \end{aligned}$$

The latter is the form taken here by condition (4) of the introduction.

The space $\mathcal{L}_\alpha$ can alternately be characterized as the class of kernels q satisfying the conditions (I) of Proposition 1.8. For $q \in \mathcal{L}_\alpha$, we set

$$(11) \quad |q|_\alpha = \|q_{11}\|_{\alpha, \infty} + \|q_1\|_{\alpha+1, \infty}.$$

By Lemma 1.3 and Proposition 1.8 it follows that

$$\|q\|_{\alpha+2, \infty} = \|1^{(2)} * q_{11}\|_{\alpha+2, \infty} \leq \frac{1}{\alpha(\alpha+1)} \|q_{11}\|_{\alpha, \infty}$$

and

$$\|q_1\|_{\alpha+1, \infty} = \|1 * q_{11}\|_{\alpha+1, \infty} \leq \frac{1}{\alpha} \|q_{11}\|_{\alpha, \infty}$$

and hence all linear combinations of $\|q\|_{\alpha+2, \infty}$, $\|q_1\|_{\alpha+1, \infty}$, and $\|q_{11}\|_{\alpha, \infty}$, which include $\|q_{11}\|_{\alpha, \infty}$ give equivalent norms on $\mathcal{L}_\alpha$. Our choice (11) above is dictated by convenience.

We remark finally that $\mathcal{L}_\alpha$ and $\mathcal{A}_\alpha$ are Banach spaces under their respective norms.

THEOREM 2.1. *If $q \in \mathcal{L}_\alpha$ then the kernel $\Gamma(q)$ defined by*

$$\Gamma(q)(x, y) = \frac{\partial^2}{\partial x \partial y} \int_0^y q(\xi + x - y, \xi) d\xi \quad (0 \leq y < x \leq 1)$$

satisfies (8), belongs to $\mathcal{A}_\alpha$ and $\|\Gamma(q)\|_{\alpha, \infty} \leq |q|_\alpha$.

Proof. Since q_1 and q_{11} are continuous on $0 \leq y + \epsilon \leq x \leq 1$ ($\epsilon > 0$) the Leibniz rule for differentiating an integral with parameter can be applied twice to $\int_0^y q(\xi + x - y, \xi) d\xi$. This gives (applying either $\partial^2/\partial x \partial y$ or $\partial^2/\partial y \partial x$)

$$\Gamma(q)(x, y) = - \int_0^y q_{11}(\xi + x - y, \xi) d\xi + q_1(x, y).$$

From this follows the continuity of $\Gamma(q)$ on $0 \leq y < x \leq 1$ and

$$\begin{aligned} \Gamma(q)(x, y) &\leq \int_0^y \frac{\|q_{11}\|_{\alpha, \infty}}{(x-y)^{1-\alpha}} d\xi + (x-y)^\alpha \|q_1\|_{\alpha+1, \infty} \\ &\leq \frac{\|q_{11}\|_{\alpha, \infty} + \|q_1\|_{\alpha+1, \infty}}{(x-y)^{1-\alpha}} = \frac{|q|_\alpha}{(x-y)^{1-\alpha}}. \end{aligned}$$

Hence $\|\Gamma(q)\|_{\alpha, \infty} \leq |q|_\alpha$. Since

$$\begin{aligned} 1 * \Gamma(q)(x, y) &= \int_y^x d\eta \left[\frac{\partial^2}{\partial \eta \partial y} \int_0^y q(\xi + \eta - y, \xi) d\xi \right] \\ &= - \int_0^y q_1(\xi + \eta - y, \xi) d\xi + q(\eta, y) \Big|_{\eta=y}^{\eta=x} \\ &= q(x, y) - \int_0^y q_1(\xi + x - y, \xi) d\xi \\ &\quad + \int_0^y q_1(\xi, \xi) d\xi - q(y, y) \end{aligned}$$

and

$$\begin{aligned} \Gamma(q) * 1(x, y) &= \int_y^x d\eta \left[\frac{\partial^2}{\partial \eta \partial x} \int_0^\eta q(\xi + x - \eta, \xi) d\xi \right] \\ &= \int_0^\eta q_1(\xi + x - \eta, \xi) d\xi \Big|_{\eta=y}^{\eta=x} \\ &= \int_0^x q_1(\xi, \xi) d\xi - \int_0^y q_1(\xi + x - y, \xi) d\xi \end{aligned}$$

we have

$$1 * \Gamma(q) - \Gamma(q) * 1 = q(x, y) - \int_y^x q_1(\xi, \xi) d\xi - q(y, y).$$

But the last two terms vanish since $q \in \mathcal{L}_\alpha$ so that (4) is satisfied by $\Gamma(q)$.

REMARK. For a kernel k of the form $k(x, y) = m(y)/m(x)$, it can be shown that the equation

$$k * \Gamma(q) - \Gamma(q) * k = q$$

is formally solved by

$$\Gamma(q)(x, y) = \frac{m(y)}{m(x)} \frac{\partial^2}{\partial x \partial y} \int_0^y q(\xi + x - y, \xi) \frac{m(\xi + x - y)}{m(\xi)} d\xi$$

provided $q(x, x) = q_1(x, x) \equiv 0$. By using this observation, results analogous to those of the present paper can be obtained for Volterra operators K with kernels k of the above type.

3. Solution of the operator equation $Q + P\Gamma(Q) = -P$. For Volterra operators P and Q with kernels $p, q \in \mathcal{L}_\alpha$ the operator equation (5) is equivalent to

$$(12) \quad q + p * \Gamma(q) = -p,$$

i.e., to the integro-differential equation

$$q(x, y) + \int_y^x p(x, \eta) \left[\frac{\partial^2}{\partial \eta \partial y} \int_0^y q(\xi + \eta - y, \xi) d\xi \right] d\eta = -p(x, y).$$

We show that, given any $p \in \mathcal{L}_\alpha$, this equation is uniquely solvable for $q \in \mathcal{L}_\alpha$.

We show in fact that, given $p \in \mathcal{L}_\alpha$, the mapping

$$(13) \quad \Gamma_p: q \rightarrow p * \Gamma(q)$$

is quasi-nilpotent on any of the spaces $\mathcal{L}_\beta$. From this follows the existence of $(I + \Gamma_p)^{-1}$ and hence the unique solvability of

$$(14) \quad q + p * \Gamma(q) = r$$

for $q \in \mathcal{L}_\beta$, given any $r \in \mathcal{L}_\beta$. The critical thing, then, is to obtain estimates of the norms of the iterates of the operator Γ_p .

LEMMA 3.1. *If $p \in \mathcal{L}_\alpha$ and $q \in \mathcal{L}_\beta$ then $p * \Gamma(q) \in \mathcal{L}_{\alpha+\beta}$ and*

$$(15) \quad |p * \Gamma(q)|_{\alpha+\beta} \leq B(\alpha, \beta) |p|_\alpha |q|_\beta.$$

Proof. We have

$$p * \Gamma(q) = \mathbf{1}^{(2)} * p_{11} * \Gamma(q) \in \mathbf{1}^{(2)} * \mathcal{L}_\alpha * \mathcal{L}_\beta \subset \mathbf{1}^{(2)} * \mathcal{L}_{\alpha+\beta} = \mathcal{L}_{\alpha+\beta}$$

and

$$(p * \Gamma(q))_1 = p_1 * \Gamma(q), \quad \text{and} \quad (p * \Gamma(q))_{11} = p_{11} * \Gamma(q).$$

Hence by Lemma 1.3

$$\|(p * \Gamma(q))_1\|_{\alpha+\beta+1, \infty} \leq B(\alpha+1, \beta) \|p_1\|_{\alpha+1, \infty} \|\Gamma(q)\|_{\beta, \infty}$$

and

$$\|(p * \Gamma(q))_{11}\|_{\alpha+\beta, \infty} \leq B(\alpha, \beta) \|p_{11}\|_{\alpha, \infty} \|\Gamma(q)\|_{\beta, \infty}.$$

Now (15) is obtained by adding the two inequalities and then using the fact that $\|\Gamma(q)\|_{\beta, \infty} \leq |q|_\beta$ and $B(\gamma, \beta) \leq B(\alpha, \beta)$ when $\gamma \geq \alpha$.

THEOREM 3.2. *If $p \in \mathcal{L}_\alpha$ then $\Gamma_p: q \rightarrow p * \Gamma(q)$ is a bounded quasi-nilpotent*

operator on $\mathcal{L}_\beta$. Hence, given $r \in \mathcal{L}_\beta$, the equation (14) is uniquely solvable for $q \in \mathcal{L}_\beta$.

Proof. An easy induction argument using Lemma 3.1 and the identity $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$ yields

$$|\Gamma_\beta^n(q)|_{n\alpha+\beta} \leq \frac{\Gamma(\alpha)^n \Gamma(\beta)}{\Gamma(n\alpha + \beta)} |p|_\alpha^n |q|_\beta.$$

Since the norms $|\cdot|_\gamma$ increase with γ , we surely have $|\Gamma_\beta^n(q)|_\beta$ dominated by the right-hand side of the above inequality. Since $\lim_{n \rightarrow \infty} \Gamma(n\alpha + \beta)^{1/n} = \infty$ the theorem follows.

4. The similarity of $J + P$ and J . On the basis of Theorems 3.2, 2.1, and Lemma 1.5 and the general considerations of the introduction we have our main result.

THEOREM A. *If $p \in \mathcal{L}_\alpha$ for some $\alpha > 0$, then the operators J and $J + P$ where*

$$J: f(x) \rightarrow \int_0^x f(y) dy$$

and

$$P: f(x) \rightarrow \int_0^x p(x, y) f(y) dy$$

are similar on $L^r(0, 1)$ ($1 \leq r \leq \infty$). This similarity $J + P \sim J$ is implemented by the operator $X = I + \Gamma(Q)$, $\Gamma(Q)$ being the Volterra operator with kernel

$$\Gamma(q)(x, y) = \frac{\partial^2}{\partial x \partial y} \int_0^y q(\xi + x - y, \xi) d\xi \quad (0 \leq y < x \leq 1),$$

where $q \in \mathcal{L}_\alpha$ is the unique solution of $q + p * \Gamma(q) = -p$.

The preceding theorem can be strengthened by a modification of a procedure used by Volterra-Pérès [9] and Kalisch [5].

Let $G: f(x) \rightarrow \int_0^x g(x, y) f(y) dy$ be a Volterra operator whose kernel satisfies

- (i) $g(x, y)$ and $g_1(x, y)$ are continuous on $0 \leq y \leq x \leq 1$.
- (ii) $g(x, x) > 0$ and $\int_0^1 g(x, x) dx = c$.
- (iii) $(d/dt)\bar{g}(t)$ and $(d/dt)\bar{g}_1(t)$ are continuous on $0 \leq t \leq 1$, where $\bar{g}(t) = g(t, t)$ and $\bar{g}_1(t) = g_1(t, t)$.
- (iv) $g_{11}(x, y)$ is continuous on $0 \leq y < x \leq 1$ and $\|g_{11}\|_{\alpha, \infty} < \infty$, where $0 < \alpha \leq 1$.

COROLLARY B. *G is similar to cJ .*

This will follow easily from the next lemmas.

LEMMA 4.1. *Let G be as above with $c = 1$, and set $n(x) = \int_0^x g(t, t) dt$. Then $S_n: f(x) \rightarrow f(n(x))$ is a bounded nonsingular operator on $L'(0, 1)$. Moreover, $H = S_n^{-1}GS_n$ is a Volterra operator whose kernel h satisfies $h(x, x) \equiv 1$ and the conditions (i) to (iv) above.*

Proof. Since $g(t, t)$ is continuous and > 0 on $0 \leq t \leq 1$, and $\int_0^1 g(t, t) dt = 1$, both n and $m = n^{-1}$ give continuously differentiable changes of variable on $[0, 1]$; $dn/dx = g(x, x)$ and $dm/dx = g(m(x), m(x))^{-1}$. Thus S_n and $S_n^{-1} = S_m$ are bounded operators on $L'(0, 1)$ (bounds $\leq \|dm/dx\|_\infty^{1/r}$ and $\|dn/dx\|_\infty^{1/r}$, respectively). Moreover, since

$$S_n^{-1}GS_n f(x) = \int_0^{m(x)} g(m(x), y) f(m(y)) dy = \int_0^x \frac{g(m(x), m(y))}{g(m(y), m(y))} f(y) dy,$$

$H = S_n^{-1}GS_n$ is a Volterra operator with kernel

$$h(x, y) = \frac{g(m(x), m(y))}{g(m(y), m(y))}$$

satisfying $h(x, x) \equiv 1$. Now

$$h_1(x, y) = \frac{g_1(m(x), m(y))}{\tilde{g}(m(y))\tilde{g}(m(x))}$$

and

$$h_{11}(x, y) = \frac{1}{\tilde{g}(m(y))} \left[\frac{g_{11}(m(x), m(y))}{\tilde{g}(m(x))^2} - \frac{g(m(x), m(y)) \frac{d\tilde{g}}{dt}(m(x))}{\tilde{g}(m(x))^3} \right].$$

In view of the above expression for h_1 , the continuity of h_1 and $d\tilde{h}_1/dt$ follows from the continuity of g_1 and $d\tilde{g}_1/dt$. Similarly, h_{11} is continuous on $0 \leq y < x \leq 1$ by the assumptions (i)-(iv) on g . To see that h_{11} satisfies the proper growth condition at the diagonal, $h_{11}(x, y) = O[(x - y)^{\alpha-1}]$, notice that in the above expression for h_{11} , only the term containing $g_{11}(m(x), m(y))$ can be unbounded near $x = y$. But by the assumption (iv) on g_{11} , $g_{11}(m(x), m(y)) = O[(m(x) - m(y))^{\alpha-1}]$ which in turn is $O[(x - y)^{\alpha-1}]$ since $x - y = n(m(x)) - n(m(y)) = \int_{m(y)}^{m(x)} g(t, t) dt$.

LEMMA 4.2. *Let H be a Volterra operator whose kernel h satisfies $h(x, x) \equiv 1$ and (i) to (iv) above and set $k(x) = \exp \int_0^x h_1(t, t) dt$. Then $M_k: f(x) \rightarrow k(x)f(x)$ is a bounded nonsingular operator on $L'(0, 1)$. Moreover, $Q = M_k^{-1}HM_k$ is a Volterra operator whose kernel q satisfies (i), (iv) and $q(x, x) \equiv 1$, $q_1(x, x) \equiv 0$.*

Proof. Since $M_k^{-1}HM_k: f(x) \rightarrow \int_0^x (k(y)/k(x))h(x, y)f(y) dy$, Q is a Volterra operator with kernel $q(x, y) = h(x, y) \exp[-\int_y^x h_1(t, t) dt]$ so that $q(x, x) = h(x, x) \equiv 1$ and

$$q_1(x, y) = [h_1(x, y) - h_1(x, x)h(x, y)] \exp \left[- \int_y^x h_1(t, t) dt \right]$$

$$q_{11}(x, y) = \left[h_{11}(x, y) - h(x, y) \frac{d\tilde{h}_1}{dt}(x) + \tilde{h}_1(x)^2 h(x, y) \right] \exp \left[- \int_y^x h_1(t, t) dt \right].$$

Thus $q_1(x, x) = h_1(x, x) - h_1(x, x)h(x, x) \equiv 0$. That the properties (i) and (iv) hold for q follows from the above expressions for q , q_1 and q_{11} and the assumptions (i) to (iv) on h .

Proof of B. Multiplying by $1/c$, G can be normalized so that $\int_0^1 g(t, t) dt = 1$. Then by the lemmas, G is similar to a Volterra operator Q whose kernel satisfies $q(x, x) \equiv 1$, $q_1(x, x) \equiv 0$, and (i) and (iv). But then the operator $P = Q - J$ has kernel $p = q - 1 \in \mathcal{L}_a$ and hence by Theorem A, $Q = J + P$ is similar to J .

5. Applications. The Volterra operator $G: f(x) \rightarrow \int_0^x g(x, y)f(y) dy$ is similar to J if, say,

$$g(x, y) = e^{\lambda(x-y)} \quad \text{where } \lambda \text{ is any complex number}$$

or if

$$g(x, y) = 1 + \frac{(x-y)^{\beta-1}}{\Gamma(\beta)} \quad \text{where } \beta \geq 2.$$

This latter example shows that J is similar to $J + J^\beta$ when $\beta \geq 2$ where J^β is the fractional integral operator,

$$J^\beta: f(x) \rightarrow \frac{1}{\Gamma(\beta)} \int_0^x (x-y)^{\beta-1} f(y) dy.$$

By a result of Kalisch [7], J is not similar to $J + J^\beta$ when $\beta < 2$. Thus Theorem A is sharp with respect to the allowable algebraic singularity of p_{11} at the diagonal.

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